In this worksheet, we will review how to deal with factorials. A **factorial** is defined, for nonnegative integers, by

$$0! = 1 \qquad \text{and} \qquad n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots n \text{ for } n \geq 1$$

Computation: Let's practice computing a few factorials:

- $0! =$ _____
- $1! =$ _____
- $2! =$ _____
- $3! =$ _____
- $4! =$ _____
- $5! =$ _____

Simplifying: Our goal will often be to simplify the ratio of factorials. To do so, we can begin by writing out all the terms:

$$\frac{7!}{5!} = \frac{\cancel{1} \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdot \cancel{5} \cdot 6 \cdot 7}{\cancel{1} \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdot \cancel{5}} = 6 \cdot 7 = 42$$

or

$$\frac{7!}{4!5!} = \frac{\cancel{1} \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdot \cancel{5} \cdot 6 \cdot 7}{(1 \cdot 2 \cdot 3 \cdot 4)(\cancel{1} \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdot \cancel{5})} = \frac{6 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{42}{24} = \frac{7}{4}$$

We can do the same with much larger factorials as well, in which we would not want to write each term:

$$\frac{100!}{98!} = \frac{1 \cdot 2 \cdot 3 \cdots 97 \cdot 98 \cdot 99 \cdot 100}{1 \cdot 2 \cdot 3 \cdots 97 \cdot 98} = \frac{\cancel{1} \cdot \cancel{2} \cdot \cancel{3} \cdots \cancel{97} \cdot \cancel{98} \cdot 99 \cdot 100}{\cancel{1} \cdot \cancel{2} \cdot \cancel{3} \cdots \cancel{97} \cdot \cancel{98}} = 99 \cdot 100 = 9900$$

While this is all fine for practice, for Calc II, we really need to simplify expressions involving some arbitrary number n. We will use "$\cdots$" notation for writing factorials involving an arbitrary n:

$$n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$$

or

$$(2n)! = 1 \cdot 2 \cdot 3 \cdots (2n-3) \cdot (2n-2) \cdot (2n-1) \cdot (2n)$$

To simplify expressions involving n, again, write down a number of terms at the start and end of the product separated with a "$\cdots$":

$$\frac{(n+2)!}{n!} = \frac{1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n \cdot (n+1) \cdot (n+2)}{1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n}$$

We can cancel just like we did with numbers:

$$\frac{(n+2)!}{n!} = \frac{\cancel{1} \cdot \cancel{2} \cdot \cancel{3} \cdots \cancel{(n-1)} \cdot \cancel{n} \cdot (n+1) \cdot (n+2)}{\cancel{1} \cdot \cancel{2} \cdot \cancel{3} \cdots \cancel{(n-1)} \cdot \cancel{n}} = (n+1)(n+2)$$

Exercises: Here is some practice for you:

1. Write each expression as a factorial:
   (a) $1 \cdot 2 \cdot 3 \cdots 18$

   (b) $1 \cdot 2 \cdot 3 \cdots (n-1) \cdot (n) \cdot (n+1)$

   (c) $1 \cdot 2 \cdot 3 \cdots (3n-1) \cdot (3n) \cdot (3n+1)$

2. Simplify the following:
   (a) $\dfrac{4!}{6!}$

   (b) $\dfrac{(3!)^2}{9}$

   (c) $\dfrac{3! \cdot 4!}{5!}$

   (d) $\dfrac{77!}{80!}$

   (e) $\dfrac{38! \cdot 3!}{39!}$

3. Simplify the following:
   (a) $\dfrac{(n+4)!}{n!}$

   (b) $\dfrac{n!}{n}$

   (c) $\dfrac{(n-1)! \cdot n!}{(n!)^2}$

   (d) $\dfrac{(n^2-1)!}{(n^2)!}$

   (e) $\dfrac{(2n)!}{(2n-2)! \cdot 2!}$